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LETTER TO THE EDITOR

The integrity bases of universal enveloping algebras and the generalised exponents

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Abstract. We obtain the generalised exponents of the Lie algebra sl(3, $\mathbb{C}$) by a method due to Kostant. Using this result we prove the hypothesis of Couture and Sharp about the existence of an integrity basis of the universal eveloping algebra of the sl(3, $\mathbb{C}$) algebra, considered as an sl(3, $\mathbb{C}$) module.

The L-module structure of the universal enveloping algebra U(L) of a semisimple Lie algebra L was shown to be completely described by the generalised exponents in the fundamental paper [1] by Kostant. This structure appears to be significant in the study of the physical models with dynamical symmetries [2].

Let $Z(L)$ denote the centre of U(L), D denote the set of dominant weights of L, V_Λ ($\Lambda \in D$) denote the simple L module with the highest weight Λ and $M(\Lambda)$ denote the multiplicity of the weight 0 of V_Λ. Then [1, 3] the $Z(L)$ module $\mathrm{Hom}_L(V_\Lambda, \mathrm{U}(L))$ has a basis formed of $M(\Lambda)$ homogeneous elements. The degrees of these homogeneous elements $m_1(\Lambda) \leqslant m_2(\Lambda) \leqslant \ldots \leqslant m_{M(\Lambda)}(\Lambda)$ are independent of the choice of the basis and are called the generalised exponents of the L module V_Λ. In the paper [1] Kostant has given the following recipe for the calculation of the generalised exponents. If $\mathrm{sl}(2, \mathbb{C}) = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$ is a principal Lie subalgebra of L [3] ($[e,f] = h$, $[h, e] = e$, $[h,f] = -f$) then the generalised exponents are the eigenvalues of h restricted to the subspace W of V_Λ the elements of which vanish under L^e (the centraliser of e in L; $L^e = \{x \in L; [x, e] = 0\}$).

We shall apply this recipe to the particular case $L = \mathrm{sl}(3, \mathbb{C})$. The fundamental weights of sl(3, $\mathbb{C}$) are denoted by Λ_1 and Λ_2. Then $D = \{\Lambda; \Lambda = k_1\Lambda_1 + k_2\Lambda_2, k_1, k_2 \text{ are integers} \geqslant 0\}$ and $D_0 = \{\Lambda \in D; \max(k_1, k_2) - \min(k_1, k_2) \equiv 0 \pmod 3\}$ is the set of all $\Lambda \in D$ for which $M(\Lambda) \neq 0$.

Theorem. Let $\Lambda \in D_0$. Then the generalised exponents of the sl(3, $\mathbb{C}$)-module V_Λ are free of multiplicity and are given by

$$m_i(k_1, k_2) = \max(k_1, k_2) + i - 1$$

where $i = 1, 2, \ldots, \min(k_1, k_2) + 1$.

Proof. Let $\{a_{ij}\}$, $i, j = 1, 2, 3$, denote a basis of the Lie algebra sl(3, $\mathbb{C}$) with the Lie brackets

$$[a_{ij}, a_{kl}] = \delta_{jk} a_{il} - \delta_{il} a_{jk}$$

and with the constraint $a_{11}+a_{22}+a_{33}=0$. The fundamental module V_{Λ_1} is a complex three-dimensional vector space. Let v_1, v_2, v_3 be a basis in V_{Λ_1} and let $e_{ij}\in \mathrm{End}(V_{\Lambda_1})$ be defined by $e_{ij}v_k=\delta_{jk}v_i$. Then the action of $\mathrm{sl}(3,\mathbb{C})$ on V_{Λ_1} is defined by

$$a_{ij}=e_{ij}-\delta_{ij}(e_{11}+e_{22}+e_{33})/3.$$

$V_{\Lambda_2}=\mathrm{Hom}_C(V_{\Lambda_1},\mathbb{C})$ and for any $\Lambda\in D$ the module V_Λ is obtained from V_{Λ_1} and V_{Λ_2} by the Cartan-Weyl method [4]. In this way the action of $\mathrm{sl}(3,\mathbb{C})$ on V_Λ is defined for any $\Lambda\in D$. The elements of V_Λ *are* labelled by the standard Young tableaux:

$$\begin{array}{c}
\overbrace{1\ \cdots\ 1}^{q_{11}}\ \overbrace{1\ \cdots\ 1\ 2\ \cdots\ 2\ \cdots\ 2}^{q_{12}-q_{11}}\ \overbrace{3\ \cdots\ 3}^{q_{13}-q_{12}}\\
\underbrace{2\ \cdots\ 2}_{q_{22}}\ \underbrace{3\ \cdots\ 3\ 3\ \cdots\ 3}_{q_{23}-q_{22}}
\end{array}$$

or by the corresponding Gel'fand-Zetlin patterns:

$$\begin{array}{ccccc}
q_{13} & & q_{23} & & 0\\
& q_{12} & & q_{22} & \\
& & q_{11} & &
\end{array}.$$

We shall take the $\mathrm{sl}(2,\mathbb{C})$ principal subalgebra of the $\mathrm{sl}(3,\mathbb{C})$ algebra to be $\mathbb{C}e+\mathbb{C}f+\mathbb{C}h$ with $e=a_{12}+a_{23}$, $f=a_{21}+a_{32}$ and $h=2a_{11}+a_{22}=e_{11}-e_{33}$. The centraliser $\mathrm{sl}(3,\mathbb{C})^e$ of the principal nilpotent element e is generated by $z_1=e$ and $z_2=e_{13}$. By definition $W=\ker z_1\cap\ker z_2$. We observe that $\ker z_2$ is generated by the elements of V_Λ which do not contain v_3 or which contain v_3 only in the antisymmetric pairs $v_1\wedge v_3$. These elements $u_{i,j}$ are labelled by the Gel'fand-Zetlin patterns

$$\begin{array}{ccccc}
k_1+k_2 & & k_2 & & 0\\
& k_1+k_2 & & j & \\
& & i & &
\end{array}$$

with $0\leqslant j\leqslant k_2\leqslant i\leqslant k_1+k_2$. From the above definitions we have evidently $hu_{i,j}=(i+j-k_2)u_{i,j}$ and

$$z_1u_{i,j}=(k_1+k_2-i)u_{i+1,j}+(k_2-j)u_{i,j+1}.$$

Any vector $w=\sum b_{i,j}u_{i,j}$ from $\ker z_2$ belongs to W if and only if $z_1w=0$. From this equation we obtain the following restrictions on the coefficients $b_{i,j}$: $b_{i,0}=0$ for $i=k_2, k_2+1,\ldots,k_1+k_2$; $b_{k_2,j}=0$ for $j=0,1,\ldots,k_2$, and

$$b_{i-1,j}=-(k_2-j+1)b_{i,j-1}/(k_1+k_2-i+1)$$

for $i=k_2+1,\ldots,k_1+k_2$ and $j=1,2,\ldots,k_2$. Because this recurrence relation conserves the value of $i+j$ it follows that all coefficients $b_{i,j}$ with $i+j$ equal to $k_2, k_2+1,\ldots,k_1+k_2$ or to $k_2,k_2+1,\ldots,2k_2$ vanish. Hence the values of $i+j$ for the non-vanishing coefficients $b_{i,j}$ are $i+j=k_1+2k_2-r$ with $r=0,1,\ldots,\min(k_1,k_2)$. For each such value of $i+j$ we obtain a vector w_r from W:

$$w_r=\sum_{s=0}^{r}(-1)^s\binom{r}{s}u_{k_1+k_2+s-r,k_2-s}.$$

Because $hw_r = (k_1 + k_2 - r)w_r$, $r = 0, 1, \ldots, \min(k_1, k_2)$, it follows that the vectors w_r are linearly independent and hence that $\dim W = \min(k_1, k_2) + 1$. The generalised exponents are exactly these eigenvalues taken in the increasing order and labelled from 1 to dim W: $m_i(k_1, k_2) = \max(k_1, k_2) + i - 1$ where $i = 1, 2, \ldots, \min(k_1, k_2) + 1$. QED

Remark 1. The result of this theorem can be obtained also as a consequence of example 1 and the first remark of the important paper [5] by Hesselink.

Remark 2. As was first observed by Couture and Sharp [6] the calculation of the generalised exponents is an easy task if we suppose the validity of their hypothesis about the existence of an integrity basis [7].

In order to formulate the Couture and Sharp hypothesis [7] about the existence of an integrity basis it is necessary to define the Cartan product $h_\Lambda h_\Omega \in \mathrm{Hom}_L(V_{\Lambda+\Omega}, \mathrm{U}(L))$ of two homomorphisms $h_\Lambda \in \mathrm{Hom}_L(V_\Lambda, \mathrm{U}(L))$ and $h_\Omega \in \mathrm{Hom}_L(V_\Omega, \mathrm{U}(L))$. Firstly we define an element $\widetilde{h_\Lambda h_\Omega}$ from $\mathrm{Hom}_L(V_\Lambda \otimes V_\Omega, \mathrm{U}(L))$ associated with any pair (h_Λ, h_Ω). This is the homomorphism which applies any vector $v_\Lambda \otimes v_\Omega \in V_\Lambda \otimes V_\Omega$ into $(h_\Lambda(v_\Lambda)h_\Omega(v_\Omega) + h_\Omega(v_\Omega)h_\Lambda(v_\Lambda))/2 \in \mathrm{U}(L)$. The Cartan product $V_{\Lambda+\Omega}$ of V_Λ and V_Ω is a simple submodule of $V_\Lambda \otimes V_\Omega$ and let $E_{\Lambda+\Omega}$: $V_{\Lambda+\Omega} \to V_\Lambda \otimes V_\Omega$ be the corresponding embedding. Then $h_\Lambda h_\Omega = h_\Omega h_\Lambda = \widetilde{h_\Lambda h_\Omega} \circ E_{\Lambda+\Omega} \in \mathrm{Hom}_L(V_{\Lambda+\Omega}, \mathrm{U}(L))$ will be called the Cartan product of h_Λ and h_Ω. Evidently degree $(h_\Lambda h_\Omega) =$ degree $h_\Lambda +$ degree h_Ω.

Following Dynkin [8] we shall say that a homomorphism h_Ω is subordinate to a homomorphism h_Λ if there exists a non-trivial homomorphism h_Ξ such $h_\Lambda = h_\Omega h_\Xi$. An integrity basis of the monoid generated by the Cartan product on the set $\{\mathrm{Hom}_L(V_\Lambda, \mathrm{U}(L)); \Lambda \in D_0\}$ is defined as the set of the homomorphisms which are subordinate to at least one homomorphism and for which there does not exist any subordinate homomorphism. Hence if h_Λ is an element of such an integrity basis then the equations $\Lambda = \Omega + \Xi$ with $\Lambda, \Omega, \Xi \in D_0$ and degree $h_\Lambda =$ degree $h_\Omega +$ degree h_Ξ do not have any non-trivial solution. When the generalised exponents, i.e. all possible values of degree h_Λ, are known for any $\Lambda \in D_0$ we can obtain the integrity basis by a thorough analysis of these equations. In this way we shall prove the validity of the Couture and Sharp hypothesis about the existence of an integrity basis [7] in the particular case of the $\mathrm{sl}(3, \mathbb{C})$ algebra. This result is the following corollary of the above theorem.

Corollary. In the case of the Lie algebra $\mathrm{sl}(3, \mathbb{C})$ there exists an integrity basis which is the union of the bases of $\mathrm{Hom}(V_{\Lambda_1+\Lambda_2}, \mathrm{U}(L))$ (the basis of which contains two elements $h_{\Lambda_1+\Lambda_2,1}$ and $h_{\Lambda_1+\Lambda_2,2}$ of degree 1 and 2 respectively), of $\mathrm{Hom}_L(V_{3\Lambda_1}, \mathrm{U}(L))$ and of $\mathrm{Hom}_L(V_{3\Lambda_2}, \mathrm{U}(L))$, with a syzygy $(h_{\Lambda_1+\Lambda_2,2})^3 = h_{3\Lambda_1}h_{3\Lambda_2}$.

Proof. From the equations $h_\Lambda = h_{\Lambda'}h_{\Lambda''}$ with $\Lambda, \Lambda', \Lambda'' \in D_0$ and where $\Lambda = \Lambda_1 + \Lambda_2$, $3\Lambda_1$, $3\Lambda_2$ we obtain the following equations $2\min(k_1', k_2') + 2\min(k_1'', k_2'') + 3n' + 3n'' = k_1 + k_2$, where $k_1 + k_2 = 2, 3, 3$ respectively. From these equations it follows immediately that either $\Lambda' = 0$ or $\Lambda'' = 0$. Hence only the trivial homomorphisms are subordinate to the homomorphisms listed in the corollary. Now we must prove that the elements of this list are subordinate to at least one homomorphism which does not belong to the list. In fact we shall prove more than that; namely that the equations

$$h_{\Lambda, m_i(\Lambda)} = h^{p_1}_{\Lambda_1+\Lambda_2,1} h^{p_2}_{\Lambda_1+\Lambda_2,2} h^{p_3}_{3\Lambda_1} h^{p_4}_{3\Lambda_2}$$

with $p_1, p_2, p_3, p_4 \in \mathbb{N} \cup 0$ and $i = 1, \ldots, \min(k_1, k_2)+1$, have a unique solution for each value of i if and only if there exists a syzygy $h^3_{\Lambda_1+\Lambda_2,2} = h_{3\Lambda_1} h_{3\Lambda_2}$. We have degree $h_{3\Lambda_1} =$ degree $h_{3\Lambda_2} = 3$. Then the above equations are equivalent with the following equations:

$$p_1 + p_2 + 3p_3 = k_1 \qquad p_1 + p_2 + 3p_4 = k_2$$

$$p_1 + 2p_2 + 3p_3 + 3p_4 = m_i(\Lambda)$$

for $\Lambda \in D_0$ and $i = 1, \ldots, \min(k_1, k_2)+1$. Because we have three equations with four unknowns the solution is not unique. In fact we have for each value of i exactly i solutions as it follows from the single equation which remains after the determination of $p_1 = \min(k_1, k_2) - i + 1$: $p_2 + 3\min(p_3, p_4) = i - 1$. Firstly we shall consider the case $i = \min(k_1, k_2)+1$ which give the minimum value 0 of p_1. In this case the minimum value of p_2 is 0 only if $\min(k_1, k_2) \equiv 0 \pmod 3$. Then $p_3 = [k_1/3]$ and $p_4 = [k_2/3]$. The maximum value of p_2 is $i - 1 = \min(k_1, k_2)$. Then $p_3 = [(k_1 - \min(k_1, k_2))/3]$ and $p_4 = [(k_2 - \min(k_1, k_2))/3]$. Hence $p_3 = p_4 = 0$ if and only if $k_1 = k_2$. If we suppose that $k_1 = k_2$ then from the fact that the generalised exponents are free of multiplicity we obtain the identities:

$$h^{k_2/3}_{3\Lambda_1} h^{k_2/3}_{3\Lambda_2} = h^{k_2}_{\Lambda_1+\Lambda_2,2}$$

for any value of k_2. Evidently, it is sufficient to impose only the case obtained for $k_2 = 3$ which is the syzygy noticed above. But the existence of this syzygy implies the unicity of the solution in the general case. Indeed, using this syzygy we can replace, when $i - 1 \not\equiv 0 \pmod 3$, the factor $(h_{3\Lambda_1} h_{3\Lambda_2})^{\min(p_3, p_4)}$ with the factor $h^{3\min(p_3,p_4)}_{\Lambda_1+\Lambda_2,2}$ (or equivalently to put $\min(p_3, p_4) = 0$) and when $i - 1 \equiv 0 \pmod 3$ we can replace the factor $h^{p_2}_{\Lambda_1+\Lambda_2,2}$ with the factor $(h_{3\Lambda_1} h_{3\Lambda_2})^{p_2/3}$ (or equivalently to put $p_2 = 0$). In both cases we have a unique solution: $p_1 = \min(k_1, k_2) - i + 1$, $p_2 = i - 1$, $\min(p_3, p_4) = 0$, $\max(p_3, p_4) = (\max(k_1, k_2) - \min(k_1, k_2))/3$, in the first case, and $p_1 = \min(k_1, k_2) - i + 1$, $p_2 = 0$, $p_3 = [(k_1 - p_1)/3]$ and $p_4 = [(k_2 - p_1)/3]$ in the second case. QED.

Remark 3. The existence of the integrity basis has been assumed without proof as an important ingredient in the description of all primitive completely prime ideals of $U(sl(3, \mathbb{C}))$ in the paper [9] by Dixmier.

Remark 4. From the remarks made in the introduction of [5] it follows that the generating functions defined by Couture and Sharp [7] are also generating functions for the generalised exponents.

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